

## **Combinatorial Tools for the Analysis of Ramified Patterns**

**J. Vannimenus<sup>1</sup> and X. G. Viennot<sup>2</sup>**

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Tree-like patterns appear in many domains of physics and the quantitative description of their morphology raises an interesting problem. To analyze their topological structure, we introduce combinatorial concepts, the bifurcation and length ratios and the ramification matrix, which generalize ideas originating in hydrogeology. Two-dimensional diffusion-limited aggregation (DLA) patterns are studied along these lines, and their statistical combinatorial properties are compared to those of random and growing binary trees and to experimental data for injection of water in clay.

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**KEY WORDS:** Combinatorial analysis; ramification; fractals; diffusion-limited aggregation; injection patterns.

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### **1. INTRODUCTION**

How can one describe *quantitatively* the “shape” of a tree, or of a ramified pattern in general? Such problems arise more and more frequently in various contexts, as image-processing equipment is becoming widely available, and there is a growing need for new, well-suited mathematical concepts. Branched patterns appear for example in recent studies on electric discharges,<sup>(1)</sup> viscous flows,<sup>(2,3)</sup> or electrochemical deposition.<sup>(4)</sup> They have been analyzed as fractal objects,<sup>(5)</sup> but trees with the same fractal dimension may have very different topologies, and other ways of characterizing important physical aspects of a tree structure have been proposed, e.g., to measure its “physical complexity” through the diffusion of a particle.<sup>(6)</sup>

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<sup>1</sup> Laboratoire de Physique Statistique de l'ENS (URA CNRS no. 731), 24 rue Lhomond, 75231 Paris Cedex 05, France and Laboratoire Louis-Neel, CNRS, 38042 Grenoble Cedex, France.

<sup>2</sup> Mathématiques et Informatique, Université Bordeaux I, 33405 Talence Cedex, France.

We propose here a combinatorial approach to study the statistical properties of the topological tree underlying these structures, and we introduce in particular a new mathematical tool, the ramification matrix.<sup>(7)</sup> Our approach is a generalization of the Horton–Strahler analysis<sup>(8,9)</sup> applied by hydrogeologists to river basins and of similar concepts used in theoretical computer science<sup>(10,11,12)</sup> and in the study of the secondary structure of biological macromolecules.<sup>(13)</sup> As an illustration, the popular diffusion-limited aggregation (DLA) model of Witten and Sander<sup>(14)</sup> is analyzed and its ramification properties are compared to those of random and growing binary trees, and to experimental data for injection patterns of water into clay in a Hele–Shaw cell.

## 2. BASIC DEFINITIONS

Let us define briefly the necessary combinatorial notions. A *binary tree* is a mathematical object consisting of vertices connected by edges: One vertex, the root, has no antecedent (or father); all other vertices have one father and are divided into internal vertices, with two sons, and external vertices (or leaves), with no son. The *order*  $k$  of a vertex is defined recursively:  $k = 1$  for a leaf,  $k = 2$  for a vertex having two leaves as sons, and in general, if the sons have orders  $k_1$  and  $k_2$ , then

$$k = \max(k_1, k_2) \quad \text{if } k_1 \neq k_2 \quad (1a)$$

$$k = k_1 + 1 \quad \text{if } k_1 = k_2 \quad (1b)$$

(for multifurcating trees the two sons with the largest orders are considered).

The *Strahler number*  $S$  of a binary tree is the maximum order of its edges (i.e., the order of its root). It is obtained inductively:  $S = 1$  for a tree reduced to one vertex, and if  $S_R$  and  $S_L$  correspond respectively to the right and left subtrees meeting at the root,

$$S = \max(S_R, S_L) \quad \text{if } S_R \neq S_L \quad (2a)$$

$$S = S_R + 1 \quad \text{if } S_R = S_L \quad (2b)$$

This parameter appears in many problems because it offers a measure of the topological size of the tree (Fig. 1)—in computer science, for instance, it gives the minimum number of registers needed to evaluate an arithmetic expression.<sup>(10,11)</sup>

A *segment* of order  $k$  is a maximal sequence of edges joining a vertex

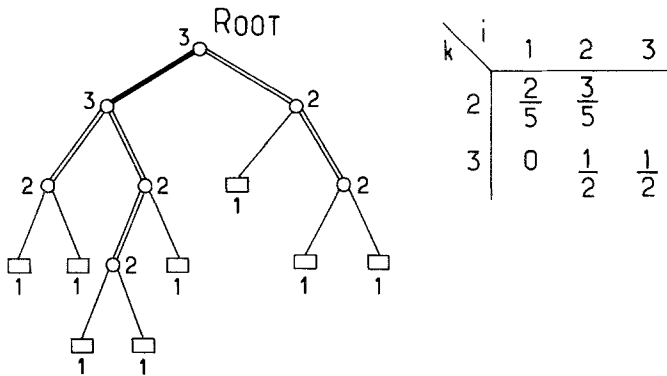


Fig. 1. A binary tree with Strahler number  $S = 3$  and its ramification matrix  $R_{i,k}$ . The vertex numbers are  $\Omega_1 = 8, \Omega_2 = 5, \Omega_3 = 2$  and the bifurcation ratios  $\beta_1 = 3/8, \beta_2 = 3$ . The order of each vertex is indicated and the “leaves” are denoted by open rectangles. The different types of lines correspond to different segment orders.

of order  $k$  to a vertex of order  $(k + 1)$ . The *bifurcation ratios*  $\beta_k$  and the *length ratios*  $\lambda_k$  are calculated from the relations

$$\beta_k = \Sigma_k / \Sigma_{k+1} \tag{3}$$

$$\lambda_k = \langle l_{k+1} \rangle / \langle l_k \rangle \tag{4}$$

where  $\Sigma_k$  is the number of segments of order  $k$  of the tree and  $\langle l_k \rangle$  is the average (physical) length of segments of order  $k$ . In applications to hydrogeology the topological tree is extracted from the map of a fluvial basin, ignoring islands: The sources of all the affluent rivers correspond to the leaves, the junctions to the vertices and the root is the mouth of the main river. Empirically,  $\beta_k \cong \beta$  and  $\lambda_k \cong \lambda$  for a given basin, independent of the order  $k$ , with  $3 \leq \beta \leq 5$  and  $1.5 \leq \lambda \leq 3$  for different river basins.<sup>(8,9)</sup>

More detailed information can be obtained by defining a new concept,<sup>(7)</sup> the *biorder*  $(k, i)$  of a vertex, as the pair of orders of its sons (the two largest ones for a multifurcating tree). If  $\omega_{k,i}$  is the number of vertices of biorder  $(k, i)$ , then the elements of the ramification matrix  $\{\mathbf{R}\}$  are defined by

$$\begin{aligned} R_{k,i} &= \omega_{k,i} / \Omega_k && \text{if } k > i \\ R_{k,k} &= \omega_{k-1,k-1} / \Omega_k && \\ R_{k,i} &= 0 && \text{if } k < i \end{aligned} \tag{5}$$

where  $\Omega_k$  is the number of vertices of order  $k$  of the tree (with these definitions the matrix  $\mathbf{R}$  is normalized:  $\sum_i R_{k,i} = 1$ ).

### 3. SIMPLE EXAMPLES AND "FAMILIES" OF TREES

It is useful to consider first several simple cases to gain a feeling for the physical content of the quantities introduced above:

(1) For the perfect binary tree all leaves lie at the same distance from the root (in physicists' language, this corresponds to a finite Bethe lattice of coordination  $z = 3$ ). The bifurcation ratios,  $\beta_k \equiv 2$ , are the smallest possible ones for a tree with  $N$  vertices, and the Strahler number,  $S = \log_2(N + 1)$ , is the largest possible one (here it is just the height of the tree). The ramification matrix  $\mathbf{R}$  has non-zero elements only on the diagonal:  $R_{k,i} = \delta_{k,i}$ .

(2) A linear comb with  $N$  teeth has only vertices of order 1 ( $\Omega_1 = N$ ) and 2 ( $\Omega_2 = N - 1$ ) and the bifurcation ratio  $\beta_1 = N$  is the largest possible one. The only non-zero elements of  $\mathbf{R}$  are

$$R_{2,1} = (N - 2)/(N - 1) \quad \text{and} \quad R_{2,2} = 1/(N - 1)$$

(3) A self-similar "fern" is obtained if every branch subdivides into  $b + 1$  branches. The bifurcation ratios are  $\beta_k \equiv b + 1$  for  $k \geq 2$  and the non-zero elements of  $\mathbf{R}$  are

$$R_{k,k} = 1/b \quad \text{and} \quad R_{k,k-1} = (b - 1)/b$$

By decorating these structures uniformly with small thorns, one introduces non-zero elements  $R_{k,1}$  in the first column. Qualitatively, large weights in the first column correspond to a "spiny" or "bushy" tree, whereas large weights along the diagonal indicate a "pruned" structure. The concept of ramification matrix gives a precise meaning to these intuitive notions.

Rather than a single tree, one is usually interested in describing an ensemble of patterns produced in a nondeterministic way. The relevant ramification matrix is then a stochastic matrix, its elements are random variables and their average value gives the probability for a vertex to have biorder  $(k, i)$ . Conversely, one can construct by a Monte Carlo procedure a typical tree representative of the "family" corresponding to a given ramification matrix. Eyrolles<sup>(12)</sup> has studied the patterns obtained in this way from different matrices, he has shown that a large variety of botanical trees can be imitated, using geometrical construction rules (branch widths and branching angles) that depend simply on the topological parameters (order and biorder). His results are in contrast with most work in this area of computer graphics, in which diversity is obtained by changing the construction rules on the same *fixed* mathematical tree, and they give a striking confirmation that the ramification matrix does contain essential information.

#### 4. ANALYSIS OF DLA AND RELATED PATTERNS

A choice system for the application of the combinatorial analysis is the diffusion-limited aggregation model of Witten and Sander, which has been successfully invoked to describe many growth phenomena.<sup>(15,16)</sup> In the standard formulation of DLA, loops are allowed but they are irrelevant for the large scale structure of the aggregates. We generated strictly loopless DLA trees on a square lattice, with  $S \leq 8$ , and their fractal dimension and appearance agreed with those of standard two-dimensional DLA. In practice, when such a theoretical model is claimed to be in good agreement with experiment, this means that the aggregates generated on a computer following the rules of the model “look similar” to the observed structures and an expert immediately recognizes two-dimensional DLA-type patterns. The brain reaches that conclusion by processing a lot of information contained in the pictures, achieving a sophisticated pattern matching task which remains a major challenge in artificial intelligence, and for which no general method exists. The computation of fractal dimensions provides a useful way to measure that similarity, but it is only a partial confirmation of a deeper intuition, and complementary mathematical tools like the ramification matrix are needed to extract more of the relevant information from the patterns.

The topological properties of DLA structures may be compared with those of some other systems:

(1) In the case of the *random binary tree*, all possible binary trees are considered, with the same weight and without any spatial constraint (they correspond to lattice animals on a Bethe lattice, i.e., in infinite dimension). The bifurcation and length ratios are given by  $\beta_k \equiv 4$ ,  $\lambda_k \equiv 2$  for a representative tree in the limit of infinite size<sup>(17)</sup> (the length of a segment is defined here as the number of its links). Their ramification matrix has the simple form of a geometric series (J. Penaud, preprint, University Bordeaux-I):

$$R_{k,i} = (1/2)^i, R_{k,k} = (1/2)^{k-1} \quad (6)$$

with a remarkable self-similarity: if all vertices of order 1 are removed (e.g., because the spatial resolution of the picture is lowered) and the remaining matrix is rescaled by a factor 2 to preserve normalization, the original form is recovered.

(2) For the *growing binary tree*, the weight of a particular aggregate is proportional to the number of different ways it can be grown: this rule favors compact configurations over filamentary ones.

(3) VanDamme et al.<sup>(18)</sup> have observed ramified patterns during the injection of water into clay in a Hele–Shaw cell, and we have studied four

of their patterns, obtained under fixed experimental conditions (Fig. 2). Here, the Strahler number was 5 or 6. The ramification analysis was carried out by hand and we checked that the identification of higher-order segments remained stable when the size of the smallest details taken into account varied. That type of stability is also realized for river basins, it insures that the results have physical meaning.

The results for these various systems are presented in Figs. 3 and 4. The length ratios  $\lambda_k$  are approximately constant for DLA and close to the random binary tree value  $\lambda_{RT}=2$ , but the bifurcation ratios  $\beta_k$  are consistently larger for small  $k$  than  $\beta_{RT}=4$  and they decrease rather rapidly when  $k$  increases (Fig. 3). The elements  $R_{k,i}$  of the DLA ramification matrix

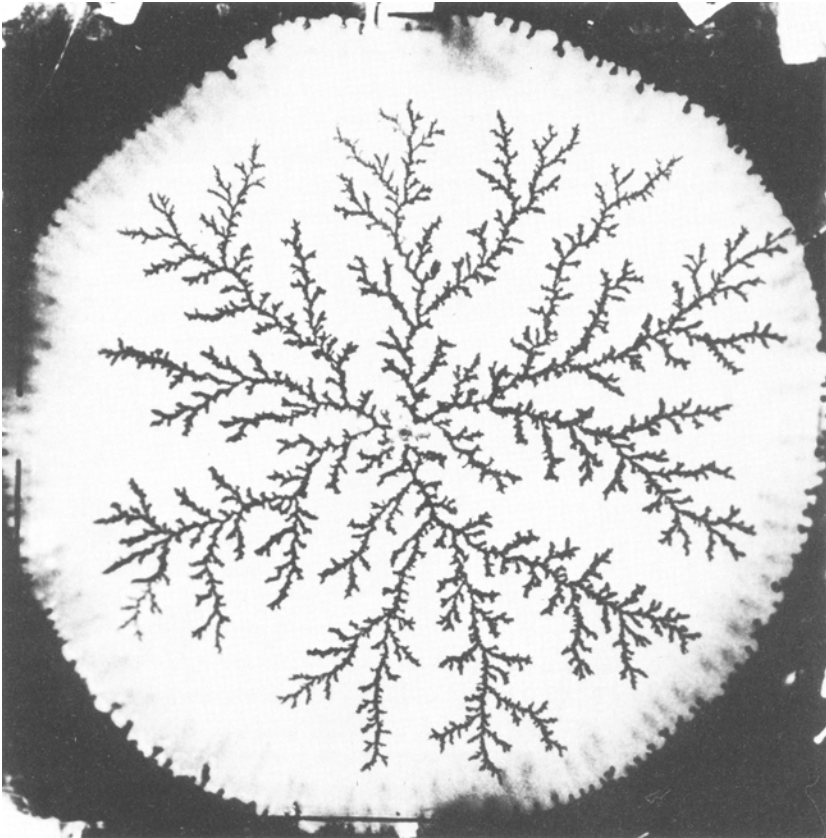


Fig. 2. One of the injection patterns of water into clay analyzed: water pressure =  $2.10^4$  Pa, clay paste at a concentration 8% per weight, in a Hele-Shaw cell of radius 25 cm and width 0.2 mm (courtesy of C. Laroche and H. VanDamme, CRSOCI Orléans).

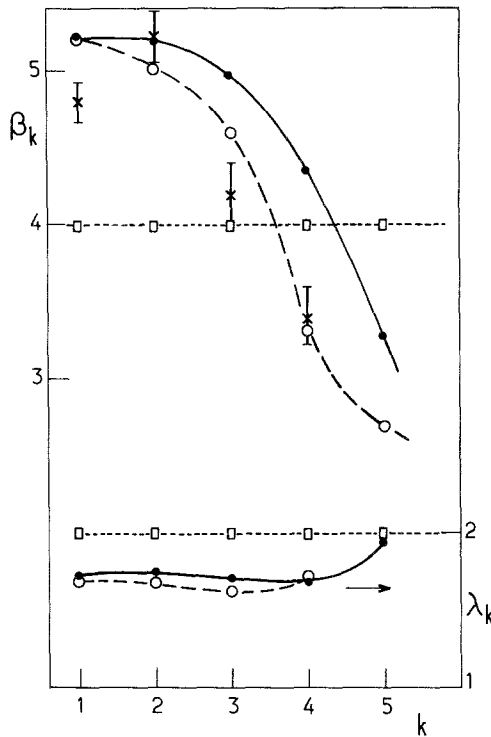


Fig. 3. Bifurcation ratios  $\beta_k$  and length ratios  $\lambda_k$  for the random binary tree ( $\square$ , exact results from ref. 17), for two-dimensional diffusion-limited aggregation, with  $10^4$  particles ( $\circ$ , 250 trees) and with  $4 \cdot 10^4$  particles ( $\bullet$ , 100 trees), and for injection patterns ( $\times$ , the bars indicate statistical fluctuations on 4 samples). The lines are just guides for the eye.

are close to the random tree values (Fig. 4), but the deviations are nonetheless significant and well outside statistical errors. In particular, the  $R_{k,i}$  decrease faster than for random trees when the index  $i$  increases. The comparison of the experimental data on injection patterns with the theoretical models shows that the best overall agreement is obtained with DLA, provided the mass of the DLA clusters is limited to  $N = 10^4$  particles, i.e., if trees of identical Strahler numbers are compared (Fig. 3). This indicates that this parameter provides a good measure of the "size" of a physical tree, and our results give a more quantitative basis to the intuitive conclusion that the DLA model can account for the morphology of two-dimensional injection patterns.

The growing tree has a markedly different behavior. Here the ramification matrix elements *increase* with the index  $i$  (Fig. 4), indicating that the structure is much less spiny. This result is rather surprising:

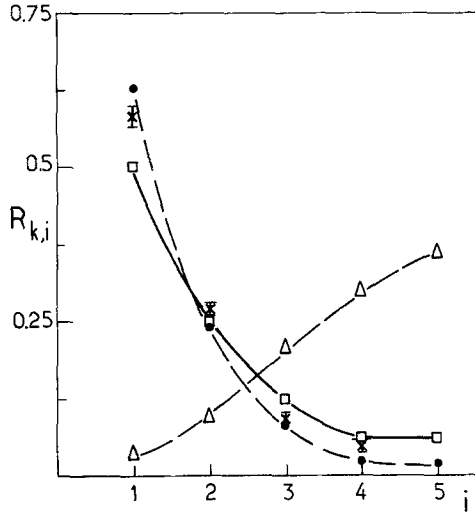


Fig. 4. Variation of the elements  $R_{k,i}$  of the ramification matrix for DLA, random binary trees and injection patterns (same symbols as in fig. 3) and for growing binary trees ( $\Delta$ , 300 samples of  $1.3 \cdot 10^5$  segments, private communication of J. G. Penaud). The data correspond to  $k = 5$ , except  $k = 4$  for injection patterns.

Indeed, the growing tree is an infinite dimension limit of DLA,<sup>(19)</sup> so one might expect similarities in their topological properties. The deep differences observed suggest that the strong constraints imposed by the two-dimensional nature of the processes play an essential role, as far as the topology is concerned, and this is confirmed by the study of another model, the Eden tree<sup>(20)</sup> (J. Vannimenus, unpublished). One may expect that in three dimensions DLA has a more complex structure, intermediate between the random and growing trees, and the comparison with the topological properties of chemical dissolution patterns<sup>(21)</sup> would be very instructive, though it raises serious practical difficulties.

## CONCLUSION

We have not attempted to *calculate* the combinatorial properties of DLA, nor to relate them to its fractal character. Our goal here is more modest and remains at a purely *descriptive* level, yet we think the new approach which we have presented should prove useful for the analysis and classification of physical branched patterns. Firstly, random trees and growing trees provide two simple examples with very different topological properties, for which detailed exact or numerical results are available for



comparison with other patterns. Secondly, our results for DLA and injection patterns show that to be meaningful such comparisons have to be made between trees of identical Strahler number, a combinatorial parameter which defines a measure of topological size. Finally, the concept of ramification matrix gives a precise meaning to such notions as “bushy” and “pruned” structures, and it offers a new quantitative way to assess the relevance of different models to experimental data. Of course, the problem of describing the shape of a tree has many facets and the combinatorial method addresses only one aspect, so that in spite of its success for the computer generation of “botanical” trees<sup>(12)</sup> different approaches may be better suited for specific systems.

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